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Transmission of continuity to the Bohr topology

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Abstract

For a locally compact Abelian (LCA) group G, let G^+ denote the group G endowed with its Bohr topology. With each piecewise affine map (defined below) α of G into another LCA group H, we show that there is associated a continuous map α^+ of G^+ into H^+ which coincides with α on a dense open subset of G^+. We study when α^+ is a homeomorphism, provided that α has this property.

These ideas are applied to investigate to what extent the group algebra of integrable functions on an LCA group G, $L^1(G)$, characterizes the group. © 1997 Elsevier Science B.V.

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1. Introduction

It is well known that every topological group G is associated to a weaker totally bounded group topology, which is the initial topology induced by the continuous homomorphism of G into its Bohr compactification, bG (we refer to Heyer [11, V, §4] for a systematic study of bG). We will denote by G^+ the group G endowed with this topology, which in the sequel will be called the Bohr topology. When G is maximally almost periodic (MAP) in the sense of von Neumann [12] (that is, for any $e_G \neq x \in G$ there is a continuous homomorphism h_x into some compact Hausdorff group K_x such that $h_x(x) \neq e_{k_x}$), G^+ is totally bounded and isomorphically embedded in bG which turns to be its Weil completion. That is, the completion of G^+ with respect to its left or

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right uniformity, since they coincide in this case (see [4, 1.4] for the definition of Weil completion, see also [20]). In case G is Abelian, the Bohr topology of G coincides with the weak topology generated by its family of continuous characters.

The Bohr topology of locally compact Abelian groups has been studied by many authors (see, for instance, [3–8,13,14,17–19]). We refer to some of them throughout the paper.

The aim of this work is the study of the continuity transmission of maps to the Bohr topology, being motivated by the following noteworthy result of Trigos-Arrieta.

Theorem 1.1 (Trigos-Arrieta [18]). *Let G and H be two locally compact Abelian groups and let $\phi: G \to H$ be a homomorphism. Then ϕ is continuous if and only if the corresponding homomorphism $\phi: G^+ \to H^+$ is continuous.*

Among other things Trigos-Arrieta applies this result to obtain non-topologically isomorphic LCA groups which are homeomorphic with respect to their Bohr topologies.

Because of Theorem 1.1 above, one could conjecture that, since homomorphisms defined between two discrete groups are always continuous for their Bohr topologies, mappings similar to homomorphisms should be, in the same vein, continuous and that they could make a natural frame to extend Trigos-Arrieta's result. Unfortunately, it is quite easy to find examples refuting this conjecture, as we now show.

In case G is a discrete group, let us denote by $G^\#$ the group G equipped with its Bohr topology.

Example 1.2. Let G be an arbitrary Abelian group with its discrete topology, and let H be any subgroup of G having infinite index. We take $e_G \neq h \in H$ fixed and define $\phi: G \to G$ as follows:

$$\phi(x) = \begin{cases} x & \text{if } x \in H, \\ x + h & \text{if } x \notin H. \end{cases}$$

Since H has infinite index in G, it has empty interior in $G^\#$, therefore ϕ is almost a translation by h, i.e., ϕ is a translation on the dense subspace $G^\# \setminus H$. Nevertheless, ϕ is not continuous on $G^\#$. Indeed, let $\{x_\delta\}_{\delta \in \Delta}$ be a net in $G \setminus H$ converging to e_G, then $\{\phi(x_\delta)\}_{\delta \in \Delta}$ converges to $h \neq e_G = \phi(e_G)$.

Thus we cannot expect an automatic transmission of continuity to the Bohr topology even for this sort of mappings. All we can say in the example above is that ϕ is Bohr-continuous on $G \setminus H$, i.e., on an open and dense subset of $G^\#$.

In the sequel we will be concerned with this kind of questions. We consider piecewise affine mappings (defined below), a class of mappings which can be seen as a natural extension of the class of group homomorphisms, and then we apply these ideas to study the following question stated by Rudin in his monograph [16, p. 95]:

Let G and H be two LCA groups such that their respective group algebras $L^1(G)$ and $L^1(H)$ are isomorphic. What can be said about the relation between G and H?

In [16] a characterization is given of when $L^1(G)$ and $L^1(H)$ are isomorphic as group algebras in terms of their dual groups. As a consequence, it is shown that $L^1(G)$ is isomorphic to $L^1(\mathbb{T})$ (from here on $\mathbb{T}$ will denote the one-dimensional torus) if and only if G is topologically isomorphic with $\mathbb{T} \times F$, for some finite Abelian group F.

Finally we consider for an LCA group G, the class $\mathcal{J}(G)$ of all groups whose group algebras are isomorphic to $L^1(G)$, and we finish giving a description of $\mathcal{J}(\mathbb{R}^n \times \mathbb{T}^m \times D)$, for n and m positive integers and D a discrete torsion-free Abelian group. Moreover we find many non-topologically isomorphic topological groups such that their group algebras are isomorphic.

On the other hand, the group algebra $L^1(G)$ characterizes G up to topological isomorphism in the realm of compactly generated torsion-free LCA groups.

2. Piecewise affine mappings

Let G be any arbitrary MAP group. We shall denote by $C_0(G)$ the coset ring of G; that is the smallest ring of subsets which contains all open cosets of G.

According to Rudin [16, p. 78], if E is a coset in G and α is a continuous mapping of E into a topological group H with α satisfying

$$\alpha(x + y - z) = \alpha(x) + \alpha(y) - \alpha(z) \quad \text{for any } x, y, z \in E,$$

then α is said to be *affine*. Notice that since E is a coset in G, it has associated a subgroup of G, say F, such that $E = x_0 + F$, for an element x_0 of E. Notice as well that α has associated a continuous homomorphism of F into H, namely the homomorphism $\overline{\alpha}$ defined by

$$\overline{\alpha}(x) = \alpha(x_0 + x) - \alpha(x_0),$$

hence Theorem 1.1 also holds for affine mappings.

If H and K are subgroups of G, and A and B are cosets of H and K, respectively, with $A \subseteq B$, then by the index of A in B we mean the index of H in K. Also, given two cosets L and J in a group G, by the index $|L : J|$, we shall understand the index of the cosets $|L : L \cap J|$ as it is done in [16], for instance.

Given a mapping α of the group G into H, α is said to be *piecewise affine* when there is a finite pairwise disjoint family

$$\{S_1, \ldots, S_n\} \subseteq C_0(G)$$

such that:

(a) $G = \bigcup_{i=1}^n S_i$.

(b) Each S_i is contained in an open coset K_i.

(c) For each $i = 1, \ldots, n$, there exists an affine mapping α_i of K_i into H satisfying

$$\alpha_{|S_i} = \alpha_{i|S_i}.$$

These concepts are taken from Rudin's book [16], although they already were used implicetely in [2] to get a representation of group algebra homomorphisms.

The maps α_i will be called the affine parts of the piecewise affine map α and we will refer to the sets $\{S_i\}_{i=1}^n$ on which α coincides with α_i as the affine representation of α. Next example is an illustration of this concept. It has been taken from [16, p. 96], see also Remark 3.1 below.

Example 2.1 (Rudin). Consider a finite abelian group $F = \{x_0, \dots, x_{n-1}\}$, then the map $\alpha_r : r + n\mathbb{Z} \longrightarrow \mathbb{Z} \times F$ defined by

$$\alpha_r(r + mn) = (m, x_r)$$

is affine for every r, $0 \leqslant r < n$, and the map $\alpha = \bigcup_{i=1}^n \alpha_i$ of $\mathbb{Z}$ onto $\mathbb{Z} \times F$ is a piecewise affine bijection with affine parts α_r, $0 \leqslant r < n$, and affine representation given by the cosets $r + n\mathbb{Z}$, $0 \leqslant r < n$.

In order to simplify arguments below, we introduce a smaller family in $C_0(G)$. Let us denote by $S(G)$ the collection of all $S \in C_0(G)$ such that

$$S = L \setminus (M_1 \cup \dots \cup M_n),$$

where L and $\{M_j\}_{j=1}^n$ are open cosets in G and each M_j is either empty or has infinite index in L. Following [16, 4.3.4] it is not difficult to show that every element of $C_0(G)$ can be represented as a finite pairwise disjoint union of members belonging to $S(G)$. We give a sketch of this fact for the sake of completeness.

Since $C_0(G)$ is the Boolean ring generated by the cosets K_i of of open subgroups H_i of G, it follows that every element of $C_0(G)$ is a pairwise disjoint union of sets in the form $P_i = (\bigcap_{j=1}^r K_{i_j}) \cap (\bigcap_{j=r+1}^n (G \setminus K_{i_j}))$. Now, it is easy to prove that the intersection of a finite family of open cosets is again an open coset, therefore, P_i can be writen as $L_i \setminus \bigcup_{j=r+1}^n K_{i_j}$ with L_i an open coset of G. Finally, in case that some K_{i_j} have finite index in L_i, we replace L_i by a finite union of translates of K_{i_j}. And we repeat this process until we get that every coset K_{i_j} is empty or has infinite index in L_i.

Next we state a technical lemma that we shall need in the sequel.

Lemma 2.2. *Let G be an Abelian topological group and let L be a coset in G. If $S = L \setminus (\bigcup_{j=1}^n M_j)$, each M_j being a closed coset in G, and $\overline{L}$ is an open coset in G then,*

$$\overline{S} = \begin{cases} \overline{L} & \text{if none of the } M_1, \dots, M_n \text{ is open,} \\ \overline{L} \setminus \left(\bigcup_{l=1}^{p} M_{j_l} \right) & \text{if } M_{j_1}, \dots, M_{j_p} \text{ are those of } M_1, \dots, M_n \text{ which are open.} \end{cases}$$

Proof. For the proof it suffices to notice that

$$\overline{S} = \overline{L} \cap \left(\overline{\bigcap_{j=1}^{n} G \setminus M_j} \right),$$

since all the sets considered are open, we obtain the desired conclusion by recalling that non-open cosets have empty interior. $\square$

From this lemma we can obtain some information about the structure of $C_0(G)$ such as the following result which generalizes [2, p. 221, Lemma].

Corollary 2.3. *Let G be an Abelian group such that*

$$G = \bigcup\{L_i\colon\ i \in I\}$$

for some cosets L_i, $i \in I$, with $|I|$ finite. If we consider the index set

$$J := \{i \in I\colon\ L_i \text{ has finite index in } G\},$$

then $G = \bigcup_{i\in J} L_i$.

Proof. We apply Lemma 2.2 to the topological group $G^\#$. If $i \in I \setminus J$ and H_i is the subgroup associated to L_i then H_i (and, hence, L_i) cannot be open in $G^\#$ otherwise

$$\frac{G^\#}{H_i} = \left(\frac{G}{H_i}\right)^\#$$

(see [3, 3.2] or [18, 2.2]) would be a totally bounded infinite discrete group, which is a contradiction. So, since every coset is closed in $G^\#$, if we consider the set $S = G \setminus \bigcup\{L_i\colon\ i \in I \setminus J\}$, an application of Lemma 2.2 yields

$$G = \overline{S}^\# \subseteq \bigcup\{L_i\colon\ i \in J\}. \qquad \square$$

Lemma 2.4. *Let G be a topological abelian group and suppose that*

$$G = \bigcup_{i\in I} S_i$$

where $S_i \cap S_j = \emptyset$ for $i \neq j$, and each S_i is of the form

$$S_i = L_i \setminus \bigcup\{M_{ij}\colon\ 1 \leqslant j \leqslant n_i\}, \quad n_i \in \mathbb{N},$$

with each L_i and M_{ij} open cosets in G, and each M_{ij} of infinite index in L_i. Let

$$J := \{i \in I\colon\ L_i \text{ has finite index in } G\}.$$

Then the family $\{L_i\colon\ i \in J\}$ forms an open partition of G.

Proof. By Corollary 2.3 we have that

$$G = \bigcup\{L_i\colon\ i \in J\}.$$

If $i \in J$, L_i is closed and of finite index in G, hence L_i is open in G^+. Thus, S_i is also open in G^+ (note that the cosets M_{ij} are closed so they are closed in the Bohr topology).

On the other hand, none of the M_{ij} can be open in G^+ since Bohr-open subgroups always have finite index. Applying now Lemma 2.2 to the sets S_i in the Bohr topology we have that $\overline{S_i}^+ = L_i$ for any $i \in J$.

Now, if we take $i \neq j$ both in J, then the sets S_i and S_j are disjoint, open and dense in the Bohr topology of the open cosets L_i and L_j, respectively, hence we have that L_i and L_j are also disjoint. $\square$

Now we can state the first result concerning the transmission of continuity of a piecewise affine map to the Bohr topology.

Theorem 2.5. *Let G and H be LCA groups and $\alpha : G \to H$ a piecewise affine map. Then there is associated to α a unique Bohr-continuous piecewise affine map $\alpha^+ : G^+ \to H^+$ such that α^+ coincides with α on an open dense member of $C_0(G^+)$.*

Proof. Since α is piecewise affine, there exists a finite pairwise disjoint subfamily $\{S_i\}_{i=1}^n \subseteq S(G)$ such that

$$S_i = L_i \setminus \bigcup\{M_{ij}\colon 1 \leqslant j \leqslant n_i\}$$

with $\{L_i\}_{i=1}^n$ and $\{M_{ij}\}_{i=1}^{n}{}_{j=1}^{n_i}$ open cosets in G with each M_{ij} of infinite index in L_i, and $\alpha_i : G \to H$ affine mappings with $\alpha_{|S_i} = \alpha_{i|S_i}$, $1 \leqslant i \leqslant n$.

Let $J = \{i \in I\colon L_i$ has finite index in $G\}$. Then by Lemma 2.4 we have that the family $\{L_i\colon i \in J\}$ forms an open partition of G^+. And now we can define

$$\alpha^+ : G^+ \to H^+ \quad \text{by } \alpha^+_{|L_i} = \alpha_i, \quad i \in J.$$

Every α_i, being affine, is continuous as a map of L_i^+ into H^+ where L_i^+ means here L_i with the topology inherited from G^+. Now it is clear that α^+ is continuous and that $S = \bigcup\{S_i\colon i \in J\}$ is an open and dense (in the Bohr topology) member of $C_0(G^+)$ such that $\alpha^+_{|S} = \alpha_{|S}$.

Notice that α^+ is uniquely determined by α, for if we had another Bohr-continuous map, say β, coinciding with α on a dense open subset of G^+, say S', then α and β would have to be the same map since they would coincide on $S \cap S'$ which is a dense subset of G^+. $\square$

Remark 2.6. Assume that α in the Theorem 2.5 is a piecewise affine homeomorphism onto, *does it follow that α^+ is also a Bohr-homeomorphism onto*? We answer negatively this question in the example below.

Example 2.7. Let $G = H = \prod_{n<\omega} K_n$ with $K_n = \{0, 1\}$, $n < \omega$, and assume that G and H have the discrete topology. Denote by λ the mapping of $2\mathbb{Z}$ onto $\mathbb{Z}$ defined by $\lambda(2n) = n$ for all $n \in \mathbb{Z}$. Let

$$L_1 = \{(x_n) \in G\colon x_{2k+1} = 0 \text{ for any } k < \omega\},$$

$$L_2 = \{(x_n) \in G\colon x_m = 0 \text{ for any } m \in \mathbb{Z} \setminus 4\mathbb{Z}\}$$

and $L_3 = G$.

Finally, let α be the piecewise affine mapping of G onto H defined by

$$\alpha((x_n)) = \begin{cases} (x_{\lambda^{-1}(n)}) & \text{for all } (x_n) \in S_1 := L_1 \setminus L_2, \\ (x_n) & \text{for all } (x_n) \in S_2 := L_2, \end{cases}$$

and for $(x_n) \in S_3 := L_3 \setminus L_1$ we define:

$$\alpha((x_n)) = (y_n) \quad \text{with } y_n = \begin{cases} 0 & \text{if } n = 2k+1, \\ x_{\lambda(n)} & \text{if } n = 2k. \end{cases}$$

It is easily verified that $\alpha(L_2) = L_2$, $\alpha(L_1 \setminus L_2) = G \setminus L_1$, and $\alpha(G \setminus L_1) = L_1 \setminus L_2$. Thus α is a piecewise affine homeomorphism onto H, since we have that $H = G = (G \setminus L_1) \cup (L_1 \setminus L_2) \cup L_2$ and

$$\{G \setminus L_1, L_1 \setminus L_2, L_2\} \subseteq S(G).$$

Since L_3 is the only coset of finite index, one has $\overline{G \setminus L_1}^{+} = G^+$. Now suppose α^+ is a homeomorphism onto. Then, by Theorem 2.5 α^+ would be a topological isomorphism of G onto the proper subgroup $\{(x_n) \in H\colon x_n = 0 \text{ if } n \text{ is odd}\}$ of H, clearly a contradiction.

Lemma 2.8. *Let G and H be topological Abelian groups and suppose $\alpha = \bigcup_{i\in I} \alpha_i$ is a piecewise affine homeomorphism of G onto H, where each α_i is affine on an open coset L_i,*

$$G = \bigcup_{i\in I} S_i,$$

with $S_i \cap S_j = \emptyset$ for $i \neq j$ and each S_i being of the form

$$S_i = L_i \setminus \bigcup\{M_{ij}\colon 1 \leqslant j \leqslant n_i\}, \quad n_i \in \mathbb{N},$$

and where each M_{ij} is an open coset of infinite index in L_i. If K_i denotes $\alpha_i(L_i)$, then each K_i is an open coset in H, each $\alpha_i : L_i \to K_i$ is an affine homeomorphism onto (i.e., its associated homomorphism $\overline{\alpha_i}$ is a topological isomorphism) and $\alpha^{-1} : H \to G$ is also a piecewise affine map.

Proof. Suppose first that for some $i \in I$, α_i is not injective, that is, we can find $x, y \in L_i$ such that $\alpha_i(x) = \alpha_i(y)$. Take now $z \in S_i$. Since $\alpha_i(z) = \alpha_i(z + x - y)$ and α_i coincides with the homeomorphism α on S_i, we have that $z + x - y \notin S_i$ and, hence, that $S_i \subseteq \cup_{j=1}^{n(i)} (y - x) + M_{ij}$, i.e.,

$$L_i \subseteq \left(\bigcup_{j=1}^{n(i)} (y - x) + M_{ij}\right) \cup \left(\bigcup_{j=1}^{n(i)} M_{ij}\right),$$

but, by Corollary 2.3, this is impossible since all the cosets M_{ij} have infinite index in L_i. We have thus proved that α_i is injective for every $i \in I$.

If we denote by N_{ij} the coset $\alpha_i(L_i \cap M_{ij})$, and by R_i the set $\alpha_i(S_i) = \alpha(S_i)$, observing that $R_i = \alpha_i(S_i) = K_i \setminus \bigcup\{N_{ij}\}$ we have that R_i belongs to $S(H)$ since for every j, $1 \leqslant j \leqslant n(i)$,

$$\alpha_i^{-1}(N_{ij}) \subseteq \bigcup_{j=1}^{n(i)} M_{ij},$$

(recall that α_i is injective) and if K_i could be covered by finitely many translates of the coset N_{ij} then M_{ij} would have finite index in L_i for some $1 \leqslant j \leqslant n(i)$.

Notice now that for any $i \in I$, K_i is an open coset in H. Indeed, S_i is open in G and α_i coincides with α on the set S_i. Since α is a homeomorphism onto H, $R_i = \alpha(S_i)$ is an open subset of H contained in K_i. This also shows that α_i is an open map onto the coset K_i. Summing up, we have obtained that α_i is a homeomorphism and that α^{-1} is a piecewise affine map with affine parts α_i^{-1}, $i \in I$. □

We see from Example 2.7 that, in order to get α^+ a homeomorphism onto, we must impose some constraints on the map α. The following result characterizes when α^+ is a Bohr homeomorphism onto.

Theorem 2.9. *Let G and H be two LCA groups and let α be a piecewise affine homeomorphism of G onto H. Let the sets $\{S_i\}_{i\in I} \subseteq S(G)$ of the form*

$$S_i = L_i \setminus \bigcup \{M_{ij}\colon\ 1 \leqslant j \leqslant n(i)\}$$

define the affine representation of α, where each M_{ij} has infinite index in L_i, and let $\alpha_i : L_i \to H$, $i \in I$, be the affine parts of α. Then a necessary and sufficient condition for α^+ to be a Bohr homeomorphism onto H^+ is that for every $i \in I$, L_i has finite index in G if and only if $\alpha_i(L_i)$ has finite index in H.

Proof. If we denote by K_i the coset $\alpha_i(L_i)$, by N_{ij} the coset $\alpha_i(L_i \cap M_{ij})$, and by R_i the set $\alpha_i(S_i) = \alpha(S_i)$, and we consider

$$J_1 = \{i\colon\ L_i \text{ has finite index in } G\} \quad \text{and} \quad J_2 = \{i\colon\ K_i \text{ has finite index in } H\},$$

then we have by Corollary 2.3 that $G = \bigcup_{i\in J_1} L_i$ and $H = \bigcup_{i\in J_2} K_i$.

Applying the generalization of Theorem 2.2 to affine mappings and Lemma 2.8, we see that α^+ is a homeomorphism onto H^+ if $J_1 = J_2$. Conversely, if α^+ maps G onto H and $i \in J_1$, then $i \in J_2$, since $\alpha^+(S_i) = \alpha(S_i) = R_i$ is open in H^+ and is contained in K_i. Also, if $i \in J_2$, we prove that $i \in J_1$ by observing that $H = \bigcup_{i\in J_1} K_i$ (because α^+ is onto) which by means of Lemma 2.4 and the inclusion just proved $J_1 \subseteq J_2$, yields that $j \in J_1$. So we have that, in case α^+ is a homeomorphism onto, $J_1 = J_2$. □

Corollary 2.10. *Let G and H be two noncompact LCA groups and α be a piecewise affine homeomorphism of G onto H with affine representation $\{S_i\}_{i\in I}$,*

$$S_i = L_i \setminus \bigcup \{M_{ij}\colon\ 1 \leqslant j \leqslant n(i)\},$$

such that, for every $i \in I$ and $1 \leqslant j \leqslant n(i)$, M_{ij} is compact. Then α^+ is a Bohr homeomorphism.

Proof. By Theorem 2.9 above we must prove that L_i has finite index in G if and only if $\alpha_i(L_i)$ has finite index in H.

Let $\alpha_i : L_i \to H$ be one of the affine parts of α. Define $R_i = \alpha(S_i)$, $K_i = \alpha_i(L_i)$ and $N_{ij} = \alpha_i(L_i \cap M_{ij})$. It is clear that $R_i = K_i \setminus \bigcup\{N_{ij}\colon\ 1 \leqslant j \leqslant n(i)\}$ since α and all the α_i are bijections by Lemma 2.8. Let us suppose that there is L_{i_0} with finite index in G such that K_{i_0} has infinite index in H.

Define $\mathcal{J} = \{i\colon 1 \leqslant i \leqslant n,\ K_i$ has finite index in $H\}$. Taking $j \in \mathcal{J}$, we know that $R_{i_0} \cap R_j = \emptyset$ since $S_{i_0} \cap S_j = \emptyset$, hence

$$R_{i_0} \subseteq H \setminus R_j \subseteq (H \setminus K_j) \cup \bigcup_{l=1}^{n(j)} N_{jl}.$$

By applying Corollary 2.3 we get $H = \bigcup_{j\in\mathcal{J}} K_j$, thus

$$R_{i_0} \subseteq \bigcup_{j\in\mathcal{J}} \bigcup_{l=1}^{n(j)} N_{jl}$$

and $\overline{R_{i_0}}^{+}$ $(= K_{i_0})$, being a closed subset of a finite union of compact subsets, is also compact. Given that K_{i_0} is homeomorphic to L_{i_0}, it follows that L_{i_0} is a compact coset of finite index in a noncompact group. This contradiction completes the proof. □

In connection with Theorem 2.9 and Corollary 2.10, it could be mentioned that it is still open the question of whether the Bohr topologies of two homeomorphic locally compact Abelian groups are homeomorphic or not. This question was posed by van Douwen [7] in the discrete case and by Trigos-Arrieta [18] in the general one. Trigos-Arrieta showed that for every LCA group G containing a closed subgroup of finite index H, topologically isomorphic to G, G^+ is topologically isomorphic to $G^+ \times \mathbb{Z}_n$, $\mathbb{Z}_n$ being the cyclic group of order $n = |G : H|$. It is easy to see that this result is a consequence of Corollary 2.10.

Despite Example 2.7 we can ensure that under certain finiteness assumptions on the involved groups α^+ is a homeomorphism between the Bohr topologies whenever α is a homeomorphism between the original locally compact topologies.

As an example we consider the case of a finitely generated Abelian group. Recall that such a group G is algebraically isomorphic to $T_G \times F_G$ where T_G is the (finite) torsion subgroup of G, and F_G is a free Abelian group. Recall as well that every subgroup of a finitely generated Abelian group is also finitely generated (cf. [10, A.27] and [9, Theorem 15.5]).

Lemma 2.11. *A subgroup $H = T_H \times F_H$ of a finitely generated Abelian group $G = T_G \times F_G$ has finite index if and only if the ranks of F_G and F_H are equal.*

Proof. Consider the chain of subgroups of G:

$$H \subseteq H + T_G \subseteq F_G + T_G = G.$$

Since $|H + T_G : H|$ is finite we have that $|G : H|$ is finite if and only if $|G : H + T_G|$ is finite. Noting that $H + T_G = F_H + T_G$ we deduce that F_G contains a subgroup, say W, isomorphic to F_H such that $|F_G : W| = |G : H + T_G|$.

Now, applying [10, A.26] we obtain that if

$$F_G = \langle x_1\rangle \times \cdots \times \langle x_n\rangle \quad \text{for some elements } x_1, \ldots, x_n,$$

then

$$F_H = \langle k_1 x_{j_1}\rangle \times \cdots \times \langle k_m x_{j_m}\rangle \quad \text{for some integers } k_1, \ldots, k_m, \text{ with } n \leqslant m.$$

It is clear that n and m are the ranks of F_G and F_H respectively, and that $|F_G : W|$ (and thus $|G : H|$) is finite if and only if these numbers are equal. □

In the following theorem, given a finitely generated group $G = T_G \times F_G$ we shall denote by $n(G)$ the rank of F_G.

Theorem 2.12. *Let G and H be two finitely generated LCA groups and let $\alpha : G \to H$ be a piecewise affine homeomorphism, then $\alpha^+ : G^+ \to H^+$ is a homeomorphism.*

Proof. As usual we have $G = \bigcup_{i=1}^n S_i$, with $S_1, \dots, S_n$ a subfamily of $S(G)$, each S_i contained in a open coset L_i, and affine mappings $\alpha_i : L_i \to H$, which by Lemma 2.8 are one to one. Recalling Theorem 2.9 we just have to prove that L_i has finite index in G if and only if $\alpha_i(L_i)$ has finite index in H. Note that we have $L_i = x_i + F_i$ for some subgroup F_i of G and $\alpha_i(L_i) = y_i + R_i$ for some subgroup R_i of H and, since α_i is injective, we have that F_i is isomorphic to R_i via the homomorphism associated to α_i.

Now suppose that F_i has finite index in G and R_i has infinite index in H. Then by Lemma 2.11 $n(F_i) = n(G)$ and $n(R_i) < n(H)$. Since F_i and R_i are isomorphic we have

$$n(G) = n(F_i) = n(R_i) < n(H).$$

Consider now R_{i_0} such that $|H : R_{i_0}|$ is finite (it must exist by Corollary 2.3 and because the cosets $\alpha_i(L_i)$, $i = 1, \dots, n$, cover H). Then we have

$$n(H) = n(R_{i_0}) = n(F_{i_0}) \leqslant n(G),$$

which yields the desired contradiction. □

3. Isomorphisms of group algebras

In this section we take notation and terminology from Rudin [16].

If G is an LCA group, $L^1(G)$ denotes the space of all complex-valued fuctions on G which are absolutely integrable with respect to the Haar measure on G, which we shall denote by m. It is well known that $L^1(G)$ is a commutative Banach algebra if multiplication is defined by convolution, that is to say, by the operation

$$(f * g)(x) = \int_G f(x - y)g(y)\,\mathrm{d}m(y) \quad \text{for every } f, g \in L^1(G),$$

and the norm

$$\|f\| = \int_G |f(x)|\,\mathrm{d}m(x), \quad \text{for every } f \in L^1(G).$$

We are now concerned with the following general question (see Rudin [16, p. 95]):

Assume that G and H are LCA groups such that their group algebras $L^1(G)$ and $L^1(H)$ are isomorphic with respect to the convolution product. What can be said about the relation between G and H?

Beurling and Helson [1] proved that if $L^1(G)$ is algebra isomorphic to $L^1(H)$ and either $\widehat{H}$ or $\widehat{G}$ are connected, then G and H are topologically isomorphic. Later in [2], Cohen gave a complete description of algebra homomorphisms between two group algebras making a strong use of the Fourier transform and idempotent measures on the dual groups from which the following theorem, which is due to Rudin is an easy consequence.

Theorem 3.1. *A necessary and sufficient condition for $L^1(G)$ and $L^1(H)$ to be isomorphic is the existence of a piecewise affine homeomorphism of $\widehat{G}$ onto $\widehat{H}$.*

We refer to the above as Cohen's theorem.

Since $\mathbb{Z}^m$ and $\mathbb{Z}^m \times F$ are piecewise affine homeomorphic (F being a finite abelian group) as it is shown in Example 2.1 above, we have one direction of the following corollary which is due to Rudin [15]:

For an LCA group G, $L^1(G)$ is algebra isomorphic to $L^1(\mathbb{T})$ if and only if $G = \mathbb{T} \oplus F$ where F is a finite Abelian group.

A systematic approach to the full problem is provided in [16, Chapter 4] where the foregoing results are to be found in Section 4.7.7.

In the sequel, we shall use the techniques developped in Section 2 above to sharpen the results quoted above.

The following remark, along with Theorem 3.4 below, are examples of how far the group algebras may be from characterizing the groups (up to topological isomorphism). Recall that if H is a subgroup of a group G, then a subset of G is called a transversal for H in G if it contains exactly one element of each coset of H in G.

Remark 3.2. If G is an LCA group containing a finite subgroup F, then $L^1(G)$ is isomorphic to $L^1((G/F) \times F)$. Indeed let Γ denote the dual group of G/F. Then Γ is an open subgroup of $\widehat{G}$ with finite index equal to $|F|$. Thus there is a transversal set of Γ in $\widehat{G}$, say T, and a bijection of F onto T, which we call β.

If we define $\alpha : \Gamma \times F \to \widehat{G}$ by

$$\alpha(x, a) = \beta(a) + x,$$

then it is clear that α is a piecewise affine homeomorphism. By using Theorem 3.1 we obtain that $L^1(G)$ is isomorphic to $L^1((G/F) \times F)$.

Notice that if F is not a direct factor of G, α cannot be an isomorphism; moreover, in many cases, for instance if G is divisible, G cannot be isomorphic to $G/F \times F$.

Next we prove a sharpening of Rudin's theorem quoted above.

Theorem 3.3. *Let G be an LCA group and D a discrete torsion-free Abelian group. Then $L^1(G)$ is isomorphic to $L^1(\mathbb{T}^m \times \mathbb{R}^n \times D)$ for some positive integers n and m if and only if G is topologically isomorphic to $\mathbb{T}^m \times \mathbb{R}^n \times D \times F$, where F is a finite Abelian group.*

Proof. Let α be the piecewise affine homeomorphism of $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$ onto $\widehat{G}$ provided by Cohen's theorem. Without loss of generality we may assume that $\alpha((0,0,e_{\widehat{d}})) = e_{\widehat{G}}$.

We have that $\alpha(\{0\} \times \mathbb{R}^n \times \widehat{D})$ is a σ-compact closed and open neighbourhood of the identity in $\widehat{G}$. Therefore $\alpha(\{0\} \times \mathbb{R}^n \times \widehat{D})$ is the connected component of the identity in $\widehat{G}$, which we denote by $C(\widehat{G})$. Thus $C(\widehat{G})$ is a σ-compact open subgroup of $\widehat{G}$.

Let α_i be an affine part of α, that is, there is $S_i \in C_0(\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D})$ and an open coset L_i in $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$ such that α_i is an affine homeomorphism of L_i into H and α_i coincides with α on the set S_i. Since S_i is closed and open in $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$, it is clear that whenever we have that $S_i \cap (\{0\} \times \mathbb{R}^n \times \widehat{D}) \neq \emptyset$, it follows that $\{0\} \times \mathbb{R}^n \times \widehat{D} \subseteq S_i$. Thus, considering the convenient restriction we have that $\{0\} \times \mathbb{R}^n \times \widehat{D}$ is topologically isomorphic to $C(\widehat{G})$. That is, $C(\widehat{G})$ is topologically isomorphic to $\mathbb{R}^n \times \widehat{D}$.

On the other hand, we claim that if L is a closed and open subgroup of $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$, then L is topologically isomorphic to either $\mathbb{R}^n \times \widehat{D}$ or $\mathbb{Z}^j \times \mathbb{R}^n \times \widehat{D}$. Indeed, if (m,r,d) is an element of L, then it must contain $\{m\} \times \mathbb{R}^n \times \widehat{D}$ since L is closed and open and $\mathbb{R}^n \times \widehat{D}$ is connected. Thus if π_1 is the natural projection of $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$ onto $\mathbb{Z}^m$, we have that $\pi_1(L) \times \mathbb{R}^n \times \widehat{D}$ is contained in L. Therefore L is isomorphic to $\pi_1(L) \times \mathbb{R}^n \times \widehat{D}$, and since $\pi_1(L)$ must be $\{0\}$ or isomorphic to $\mathbb{Z}^j$ for some $j \leqslant m$ (see [10, A.26]), our claim is proved. Thus, L has finite index if and only if $j = m$.

Suppose now that for every coset L_i in the affine representation of α with finite index in $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$, the corresponding coset $\alpha_i(L_i)$ has infinite index in $\widehat{G}$, and consider a coset L_{i_0} in the affine representation of α of finite index. Then, since the homomorphisms associated to each of the affine parts of α are isomorphisms, we have that the subgroup B_{i_0} associated to $\alpha_{i_0}(L_{i_0})$, is isomorphic to $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$. But the cosets $\alpha(L_i)$ $(i \in I)$ of the affine representation of α cover $\widehat{G}$ so, by Corollary 2.3, we know that

$$\widehat{G} = \bigcup_{j \in J} \alpha_j(L_j),$$

where $J = \{i \in I\colon\ \alpha_i(L_i)$ has finite index in $\widehat{G}\}$, thus B_{i_0} is contained in $\bigcup_{j\in J} \alpha_j(L_j)$, and this means that for some $j_1 \in J$, $\alpha_{j_1}(L_{j_1})$ is of finite index in B_{i_0}. But, for any $j \in J$, we have that $\alpha_j(L_j)$ is isomorphic to $\mathbb{Z}^{m_j} \times \mathbb{R}^n \times \widehat{D}$, with $m_j < m$, and its index in B_{i_0}, which is isomorphic to $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$, cannot be finite. We conclude that there is at least one L_i with finite index in $\mathbb{R}^n \times \mathbb{Z}^m \times \widehat{D}$, such that $\alpha_i(L_i)$ has also finite index in $\widehat{G}$.

Hence $\widehat{G}$ contains a closed and open subgroup of finite index which is isomorphic to $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$. Thus, there is a topological isomorphism β of $\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D}$ into $\widehat{G}$ and we have that the quotient group $(\mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D})/\{0\} \times \mathbb{R}^n \times \widehat{D}$, which is isomorphic to $\mathbb{Z}^m$, is topologically isomorphic to a subgroup of $\widehat{G}/C(\widehat{G})$ of finite index.

Hence $\widehat{G}/C(\widehat{G})$ is finitely generated, and thus isomorphic to $\mathbb{Z}^l \times F$, with F a finite Abelian group. Since $\mathbb{Z}^m$ has finite index, it follows from Lemma 2.11 that $\widehat{G}/C(\widehat{G})$ is isomorphic to $\mathbb{Z}^m \times F$.

Now since $C(\widehat{G})$ is open in $\widehat{G}$, we have that $\widehat{G}$ is topologically isomorphic to $C(\widehat{G}) \times \widehat{G}/C(\widehat{G})$ ([10, 24.45]), i.e., $\widehat{G}$ is topologically isomorphic to $\mathbb{R}^n \times \widehat{D} \times \mathbb{Z}^m \times F$, taking duals we obtain the desired result.

Conversely, suppose now that we have a group G topologically isomorphic to $\mathbb{T}^m \times \mathbb{R}^n \times D \times F$ where m and n are positive integers, D is a discrete torsion-free Abelian group and F is a finite Abelian group. Then $\widehat{G}$ is topologically isomorphic to $\mathbb{Z}^{m-1} \times \mathbb{R}^n \times \widehat{D} \times (F \times \mathbb{Z})$. As it is shown in Example 2.1 above there exists a piecewise affine homeomorphism of $\mathbb{Z}$ onto $\mathbb{Z} \times F$, consequently $\mathbb{Z}^{m-1} \times \mathbb{R}^n \times \widehat{D} \times (F \times \mathbb{Z})$ and $\mathbb{Z}^{m-1} \times \mathbb{R}^n \times \widehat{D} \times \mathbb{Z}$ $(= \mathbb{Z}^m \times \mathbb{R}^n \times \widehat{D})$ are piecewise homeomorphic. Applying Theorem 3.1 we obtain that $L^1(G)$ is isomorphic to $L^1(\mathbb{T}^m \times \mathbb{R}^n \times D)$. □

The following result shows that, in the opposite direction, $L^1(G)$ characterizes G in the class of torsion-free compactly generated topological groups.

Theorem 3.4. *Let G and H be two torsion-free, compactly generated LCA groups. Then $L^1(G)$ isomorphic to $L^1(H)$ if and only if G is topologically isomorphic to H.*

Proof. Necessity is obvious.

In order to prove the sufficiency, we notice that Theorem 3.1 asserts the existence of a piecewise affine homeomorphism α of $\widehat{G}$ onto $\widehat{H}$. Hence we have that $\widehat{G} = \bigcup_{i=1}^{n}\{S_i\}$, for a pairwise disjoint family $\{S_i\}_{i=1}^{n}$ contained in $S(G)$, such that $S_i = L_i \setminus \bigcup\{M_{ij}\}_{j=1}^{n(i)}$ for some open cosets $\{L_i\}_{i=1}^{n}$ and $\{M_{ij}\}_{j=1}^{n(i)}{}_{i=1}^{n}$ in G, and affine mappings $\alpha_i : L_i \to H$ each of them coinciding with α on the sets S_i. By Lemma 2.8 the homomorphisms asociated to α_i are topological isomorphisms.

If $J = \{i \in I\colon\ L_i$ has finite index in $\widehat{G}\}$, by Corollary 2.3 we know that

$$\widehat{G} = \bigcup\{L_i\colon\ i \in J\}.$$

On the other hand G and H are compactly generated LCA groups and, thus they are topologically isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m \times K_1$ and $\mathbb{R}^p \times \mathbb{Z}^q \times K_2$ respectively, for some integers n, m, p and q and for some compact groups K_1 and K_2. Hence $\widehat{G}$ and $\widehat{H}$ are, respectively, topologically isomorphic to $\mathbb{R}^n \times \mathbb{T}^m \times D_1$ and $\mathbb{R}^p \times \mathbb{T}^q \times D_2$, with D_1 and D_2 discrete groups. Since G and H are torsion-free, K_1 and K_2 will be as well, whence D_1 and D_2, being duals of compact torsion-free groups, are divisible [10, Theorem 24.23]. Summing up, we have that $\widehat{G}$ and $\widehat{H}$ are divisible groups.

Now $\widehat{G}$, being divisible, does not admit proper subgroups of finite index; therefore there must be $i_0 \in J$ such that $L_{i_0} = \widehat{G}$. We have, thus, a topological isomorphism, namely, the homomorphism $\overline{\alpha_{i_0}}$ of $\widehat{G}$ into $\widehat{H}$ associated to α_{i_0}. Furthermore, $\alpha_{i_0}(L_{i_0})$ is open in $\widehat{H}$ since S_{i_0} is open in $\widehat{G}$ and α coincides with α_{i_0} on the set S_{i_0} by Lemma 2.4. Thus the range of $\overline{\alpha_{i_0}}$ is an open subgroup of $\widehat{H}$ and, hence its connected component of the identity coincides with the component $C(\widehat{H})$ of $\widehat{H}$.

The map $\overline{\alpha_{i_0}}$ is a topological isomorphism, so it maps the component of $\widehat{G}$ onto the component of its image which is the component of $\widehat{H}$. That is

$$\overline{\alpha_{i_0}}(\mathbb{R}^n \times \mathbb{T}^m \times \{0\}) = \mathbb{R}^p \times \mathbb{T}^q \times \{0\}$$

and thus by [10, Corollary 9.13] we obtain that $n = m$ and $m = q$.

As a consequence of this, D_1 (which is isomorphic to $\widehat{G}/(\mathbb{R}^n \times \mathbb{T}^m \times \{0\})$) is isomorphically embedded in D_2 (which is isomorphic to $\widehat{H}/\overline{\alpha_{i_0}}(\mathbb{R}^n \times \mathbb{T}^m \times \{0\})$).

By exchanging the roles between $\widehat{G}$ and $\widehat{H}$ we get, repeating the same arguments, that D_2 is isomorphically embedded in D_1.

But D_1 and D_2 are discrete divisible groups each isomorphically embedded in the other, hence they are topologically isomorphic. Thus $\widehat{G}$ is topologically isomorphic to $\widehat{H}$ and, consequently, G and H are topologically isomorphic as well. □

Remark 3.5. With a similar but somewhat more involved argument, it can be proved that $L^1(G)$ characterizes the group G in the class of all torsion-free groups which are of the form $\mathbb{R}^n \times K \times D$ with K a compact Abelian group and D a discrete Abelian group.

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